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Shell-model matrix elements in the neutron–proton quasi-spin formalism using vector coherent states

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Abstract. Vector coherent state theory for the O(5) group is used to obtain a general formula for the shell-model matrix elements of any isoscalar two-body interaction in a j-shell of neutrons and protons. Explicit results for arbitrary T, j and even n are given for states with seniority not exceeding four.

1. Introduction

The seniority classification of states in a single j-shell of neutrons and protons is most simply described by the extended quasi-spin group O(5), or Sp(4), whose generators are the three $J = 0$, $T = 1$ pair operators $A^\dagger$, the three pair destruction operators, the three isospin operators and the number operator, most conveniently taken as $H_1 = (n - 2\Omega)/2$ where $\Omega = j + 1/2$. H_1 measures the particle number relative to the middle of the shell. The irreducible representations of O(5) are denoted (ωt) where ω is related to the seniority v by $\omega = \Omega - v/2$ and t is called the reduced isospin. Physically, v is the number of unpaired nucleons and t is their isospin. The complete set of states within (ωt) is then formed by adding any number of pairs with products of the operators $A^\dagger$. The Pauli principle ensures that the set is finite. Naturally, we need a classification of states within (ωt) in which the nucleon number n and the total isospin T are definite. These two quantum numbers correspond to the sub-group SU(2) × U(1) of O(5). For most states, the two labels n and T are not sufficient for a unique classification within (ωt) and there is no group contained in O(5) and containing SU(2) × U(1) which can be used to provide additional labels. The problem of introducing a complete orthonormal classification is therefore a difficult one which has attracted much attention in the past [1–4]. It has also acquired new interest through the mapping [5] from the shell-model to an isospin invariant form, IBM3, of the boson model. The advent of vector coherent state theory [6–10], has provided a new line of attack on the problem of classification within O(5).

The main purpose of this paper is to derive a general formula for the matrix elements of an isoscalar two-body interaction and to give results for states up to seniority $v = 4$ with even n. In particular, this displays the dependence of the matrix elements on the conserved quantum numbers n and T. A knowledge of this dependence will lead, in a later paper, to a derivation of the nT-dependence of the IBM3 Hamiltonian. In separate appendices we also discuss the relative merits

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of two different orthonormal O(5) bases and give the K-matrix elements for the three-fold states of the representation $(\omega 2)$. In general terms, the derivation of the nT-dependence of the shell-model matrix elements is equivalent to the calculation of O(5) Wigner coefficients, most of which are not yet known. The method used here is direct and exhibits clearly the effect of multiplicities in the Clebsch–Gordan reductions.

2. The coherent state formalism

This section contains a brief résumé of the use of vector coherent states in the O(5) classification problem, which is more fully described elsewhere [11, 12]. We attempt to give a simple and concise treatment. The vector coherent state is defined as

$$|\boldsymbol{z}, \omega t m_t\rangle = \mathrm{e}^{\boldsymbol{z}^* \cdot \boldsymbol{A}^\dagger}|\omega t m_t\rangle \tag{1}$$

where $|\omega t m_t\rangle$ is a member of the $(2t+1)$-dimensional space of states of full seniority $n = v = 2(\Omega - \omega)$ and isospin t and $\boldsymbol{z} = (z_1, z_2, z_3)$ is a vector of complex variables. Precisely, we use $\boldsymbol{A}^\dagger = \sqrt{\Omega/2}(\boldsymbol{a}^\dagger \times \boldsymbol{a}^\dagger)^{J=0,T=1}$. Each shell-model state $|\psi\rangle$ in the O(5) representation (ωt) has its z-space representatives

$$\psi_{\omega t m_t}(\boldsymbol{z}) = \langle \boldsymbol{z}, \omega t m_t|\psi\rangle \tag{2}$$

and the complete z-space state, distinguished by a round bracket, is written

$$|\psi) = K^{-1} \sum_{m_t} \psi_{\omega t m_t}(\boldsymbol{z})|\omega t m_t) \tag{3}$$

where $|\omega t m_t)$ denotes a $(2t+1)$-dimensional vector space spanned by m_t and mapped onto the full seniority states $|\omega t m_t\rangle$. The choice of the operator K is described below.

To keep the notation simple, consider that part of an operator V which transforms from (ωt) to $(\omega' t')$. Clearly any operator may be broken down into such parts. The corresponding operator $\gamma(V)$ in z-space is defined by

$$\gamma(V)|\psi) = K^{-1} \sum_{m'_t} \langle \boldsymbol{z}, \omega' t' m'_t|V|\psi\rangle|\omega' t' m'_t) \tag{4}$$

which is consistent with equations (2) and (3).

It is convenient to use the Bargmann scalar product [11] in z-space so that $\boldsymbol{z}$ and $\boldsymbol{\nabla}$ are adjoint. The mapping defined in equations (2)–(4) is not then unitary if K is the unit operator but may be made unitary by a suitable choice for K. Since the states of a given (ωt) are generated by the pair creation operators, it is sufficient to determine K from the unitarity condition on $\boldsymbol{A}^\dagger$, namely $\gamma(\boldsymbol{A}^\dagger) = \gamma(\boldsymbol{A})^\dagger$. In fact this leads [11] only to an equation for $KK^\dagger$ and K is found by diagonalizing $KK^\dagger$ before taking the square root.

It is easy to construct a convenient orthonormal basis in z-space,

$$|(\omega t) p T_p T M_T) = \left[Z^{(p0)}_{T_p}(\boldsymbol{z})|\omega t)\right]_{TM_T} \tag{5}$$

where $p=(n-v)/2$ is the number of pairs and the set of functions Z_{T_p} with isospin $T_p = p, p-2, \ldots, 1$ or 0 spans the representation $(p\,0)$ of the group U(3) in the three components of $\boldsymbol{z}$. The function Z_{T_p} is a homogeneous polynomial of order p in the components of $\boldsymbol{z}$ and is the Bargmann form of the three-dimensional harmonic oscillator wavefunction. The square bracket in equation (5) denotes vector coupling of T_p to t to give a resultant T but we follow Hecht's convention for the order of coupling tT_p to simplify phase factors at a later stage.

The matrix $KK^\dagger$ is not generally diagonal in the basis (5) but we may introduce the basis of eigenvectors of $KK^\dagger$ which are related to the basis (5) by a unitary matrix U,

$$|(\omega t)piTM_T) = \sum_{T_p} U^{-1}_{T_p i}|(\omega t)pT_pTM_T) \tag{6}$$

with

$$KK^\dagger|(\omega t)piTM_T) = \lambda_i|(\omega t)piTM_T). \tag{7}$$

K may also be taken diagonal in this basis with values $\sqrt{\lambda_i}$.

To construct the shell-model bases corresponding to the z-space bases (5) and (6) we see from equation (4) that $\gamma(\boldsymbol{A}) = K^{-1}\boldsymbol{\nabla}K$ so that $\gamma(\boldsymbol{A}^\dagger) = K^\dagger \boldsymbol{z}(K^\dagger)^{-1}$ which implies that

$$\gamma(Z^{(p0)}_{T_p}(\boldsymbol{A}^\dagger)) = K^\dagger Z^{(p0)}_{T_p}(\boldsymbol{z})(K^\dagger)^{-1}. \tag{8}$$

If K is normalized to be unity when acting on a state of full seniority, equations (3), (4) and (8) lead to the correspondence

$$\left[Z^{(p0)}_{T_p}(\boldsymbol{A}^\dagger)|\omega t\rangle\right]_{TM_T} \to K^\dagger|(\omega t)pT_pTM_T) \tag{9}$$

$$= \sum_i K^\dagger_{iT_p}|(\omega t)piTM_T) \tag{10}$$

$$= \sum_{T'_p} K^\dagger_{T'_pT_p}|(\omega t)pT'_pTM_T). \tag{11}$$

On the left-hand side of equation (9) is the non-orthonormal shell-model basis with p pairs coupled to isospin T_p which is coupled to the full seniority state in the same order tT_p as in equation (5). Explicitly, the matrix elements of K are given in terms of the solutions of the eigenvalue problem by

$$K^\dagger_{iT_p} = \sqrt{\lambda_i}\,U_{iT_p} \tag{12}$$

$$K^\dagger_{T'_pT_p} = \sum_i U^{-1}_{T'_p i}\sqrt{\lambda_i}\,U_{iT_p}. \tag{13}$$

The K-matrix of equation (13) is Hermitian. By using the inverse matrices, equations (10) and (11) give the desired orthonormal shell-model bases, denoted on the left by angled brackets, which correspond to the z-space bases (5) and (6),

$$|(\omega t)piTM_T\rangle = \sum_{T_p}(K^\dagger)^{-1}_{T_p i}\left[Z^{(p0)}_{T_p}(A^\dagger)|\omega t\rangle\right]_{TM_T} \to |(\omega t)piTM_T) \tag{14}$$

$$|(\omega t)pT_pTM_T\rangle = \sum_{T'_p}(K^\dagger)^{-1}_{T'_pT_p}\left[Z^{(p0)}_{T'_p}(A^\dagger)|\omega t\rangle\right]_{TM_T} \to |(\omega t)pT_pTM_T). \tag{15}$$

The basis (14) is called the 'most natural' in [11] since it corresponds to the eigenvectors of $KK^\dagger$ but in appendix A we show that there is also some merit in the basis (15).

The unitarity of the mapping (3) enables us to write the shell-model matrix-element of any operator V in terms of a z-space matrix element of $\gamma(V)$,

$$\langle\varphi|V|\psi\rangle = (\varphi|\gamma(V)|\psi). \tag{16}$$

To calculate the right-hand side of equation (16) it is best to separate out the K-matrix factors by defining $\Gamma(V) = K\gamma(V)K^{-1}$ so that $\Gamma(V)$ is the simpler mapping defined by equations (3) and (4) with the K^{-1} factors removed. Furthermore, $\Gamma(V)$ is more simply evaluated in the T_p basis (5) rather in the i-basis (6). Hence, in the two bases (14) and (15) respectively, we have

$$\langle i'|V|i\rangle = \sum_{\tilde{T}_p\tilde{T}'_p} K^{-1}_{i'\tilde{T}'_p}K_{\tilde{T}_p i}(\tilde{T}'_p|\Gamma(V)|\tilde{T}_p) \tag{17}$$

$$\langle T'_p|V|T_p\rangle = \sum_{\tilde{T}_p\tilde{T}'_p} K^{-1}_{T'_p\tilde{T}'_p}K_{\tilde{T}_pT_p}(\tilde{T}'_p|\Gamma(V)|\tilde{T}_p). \tag{18}$$

For brevity we have omitted the labels $(\omega t)pTM_T$ from the right-hand side in these matrix elements and the corresponding primed labels from the left. The matrix elements of K again come from equations (12) and (13). In the next section we describe the calculation of the matrix elements of $\Gamma(V)$ required on the right-hand side of both of equations (17) and (18) for a general isoscalar two-body interaction.

3. The matrix elements of an isoscalar two-body interaction

The general two-body shell-model interaction may be analysed [1]

$$V = V^{(00)} + V^{(11)} + V^{(20)} + V^{(22)} \tag{19}$$

into parts with definite O(5) character. Since H conserves particle number and isospin, all parts have $H_1 = T = 0$. The second part has a very simple explicit form

$$V^{(11)} = \left\{3\sum_{\text{even } J}(2J+1)E_J + \sum_{\text{odd } J}(2J+1)E_J\right\}H_1/2\Omega \tag{20}$$

where $E_J = \langle J|V|J\rangle$ is the two-body shell-model energy. Hence the contribution from $V^{(11)}$ depends only on the nucleon number through H_1 and is independent of all other labels for the many-nucleon system. Since $V^{(00)}$ is an O(5) scalar it is obviously diagonal in (ωt) and a multiple $\langle\omega t|V^{(00)}|\omega t\rangle$ of the unit matrix within (ωt) where $|\omega t\rangle$ denotes the state of full seniority $n = v = 2(\Omega-\omega)$ with $p = 0$ and $T = t$. The value of this matrix element is discussed in section 4. The purpose of this section is to derive a general formula relating the many-nucleon matrix elements of the last two parts of V to a small number of reduced matrix elements. (Because O(5) is not simply reducible, the Wigner–Eckart theorem does not lead to a single reduced matrix element for each part.) Results for low even seniorities, up to $v = 4$, are given in appendix C. Explicit forms for the reduced matrix elements are obtained in section 4.

Assuming that the matrices K are known, see appendix D, our objective is to find the z-space matrix elements

$$((\omega' t')p'T_p'TM_T|\Gamma(V)|(\omega t)pT_pTM_T) \tag{21}$$

on the right-hand side of equations (17) and (18) for the last two parts of V in equation (19). For some chosen (ωt) and $(\omega' t')$ the operator $\Gamma(V)$ is conveniently analysed into multipoles or expressed in an $m'm$ basis

$$\Gamma(V) = \sum_r g^{(r)}\cdot G^{(r)} = \sum_{m_t'm_t} g_{m_t'm_t}|\omega' t' m_t')(\omega t m_t| \tag{22}$$

where $G^{(r)}$ is independent of z and is defined by

$$(\omega' t'\|G^{(r)}\|\omega t) = 1 \tag{23}$$

while $g^{(r)}$ is a function of z and ∇ and the two forms of g are related by

$$g_\rho^{(r)} = (2r+1)\sum_{m_t'm_t}(-1)^{m_t-t'} g_{m_t'm_t}\begin{pmatrix} t & t' & r \\ m_t & -m_t' & -\rho \end{pmatrix}/\hat{t}'. \tag{24}$$

Some elementary Racah algebra leads to

$$\begin{aligned}((\omega' t')p'T_p'TM_T|\Gamma(V)|(\omega t)pT_pTM_T) \\ = (-1)^{t+T_p'+T}\hat{t}'\hat{T}_p'\sum_r\begin{Bmatrix} t' & t & r \\ T_p & T_p' & T \end{Bmatrix}(p'T_p'\|g^{(r)}\|pT_p)\end{aligned} \tag{25}$$

where the matrix element on the right refers only to the functions Z in equation (5).

Recall from section 2 that Γ is defined through equations (3) and (4) with K removed so that from equation (3),

$$\Gamma(V)|\psi) = \sum_{m_t'm_t}\psi_{\omega t m_t}(z)|\omega' t' m_t') \tag{26}$$

while from equation (4),

$$\Gamma(V)|\psi) = \sum_{m_t'}\langle z,\omega' t' m_t'|V|\psi\rangle|\omega' t' m_t'). \tag{27}$$

Comparison of equations (26) and (27) gives the defining equation for the operator g, where we have also used equation (1),

$$\langle \omega' t' m'_t | \mathrm{e}^{z\cdot A} V | \psi \rangle = \sum_{m_t} g_{m'_t m_t} \psi_{\omega t m_t}(z). \tag{28}$$

Here, $A_q = (-1)^q (A^\dagger_{-q})^\dagger$ and $z_q = (-1)^q (z^*_{-q})^*$.

Our task is to evaluate the shell-model matrix element on the left-hand side of this equation for the last two parts of V in equation (19). The first step is to note that $V^{(2s)}$ with $s = 0$ or 2 is a member of a standard irreducible O(5) tensor operator $V^{(2s)}_{H_1 T}$ with $H_1 = T = 0$. Commutation with the generators A_q will therefore produce other members of the tensor operator with known coefficients,

$$\left[V^{(2s)}_{H_1 T}, A^\dagger \right]^{(u)} = -\langle (2s) H_1 + 1 u \| A^\dagger \| (2s) H_1 T \rangle V^{(2s)}_{H_1+1 u} \tag{29}$$

where the notation on the left implies the vector-coupled commutator. Equation (29) follows the convention of Hecht [12] who also gives a procedure for calculating the reduced matrix elements of $A^\dagger$. Using equation (29) we may write

$$\mathrm{e}^{z\cdot A} V^{(2s)}_{00} = \sum_{mc} a_{mc} \left(V^{(2s)}_{-mc} \cdot z^{m(c)} \right) \mathrm{e}^{z\cdot A}. \tag{30}$$

The sum over $m \leqslant 2$ and c is very limited and the full set of coefficients a_{mc} is given in table 1. The notation $z^{m(c)}$ denotes m factors of z coupled to isospin c. Using equation (30) and introducing a complete set of intermediate states, the left-hand side of equation (28) becomes

$$\sum_{mc\tau\mu T'_q T_q} a_{mc} \langle \omega' t' m'_t | z^{m(c)} \cdot V^{(2s)}_{-mc} | \omega t q T'_q \tau \mu \rangle \langle \omega t q T_q \tau \mu | \mathrm{e}^{z\cdot A} | \psi \rangle (K K^\dagger (\omega t q \tau))^{-1}_{T'_q T_q}. \tag{31}$$

Because of their simple structure, we have chosen the intermediate states to be the non-orthonormal set on the left of equation (9). They are not to be confused with the orthonormal set on the left-hand side of equation (15). The $KK^\dagger$ factor also comes from equation (9), being the inverse of the norm matrix which is necessary in the resolution of the identity in a non-orthonormal basis. The number q of pairs in the intermediate state is given by $q = m + \omega - \omega'$, since H_1 is additive with value $-\omega$ in states of full seniority in (ωt). It is always possible to take $\omega' \geqslant \omega$ so that q can never exceed 2.

Table 1. The coefficients a_{mc} of equation (30).

	$m = c = 0$	$m = c = 1$	$m = 2, c = 0$	$m = 2, c = 2$
(20)	1	$\sqrt{5/3}$	$-\sqrt{5/2}$	0
(22)	1	$-\sqrt{8/3}$	0	$\sqrt{2}$

Next we insert the explicit form for the intermediate states from equation (9), through the functions $Z(A^\dagger)$. Then, from equation (29), we may write

$$\left(V^{(2s)}_{-mc} \times Z^{(q0)}_{T'_q}(A^\dagger)\right)^{(u)} = b(mcqT'_q u)V^{(2s)}_{q-m\ u} + (\text{terms with } A^\dagger \text{ on the left}) \quad (32)$$

where the second term on the right makes no contribution when used in equation (31). The full set of required coefficients b is given in table 2. With the help of the identity

$$A\mathrm{e}^{z\cdot A} = \nabla \mathrm{e}^{z\cdot A} \quad (33)$$

and some rearrangement of the angular momentum coupling, the expression (31) then reduces to

$$\begin{aligned}\sum_{m_t ucmx\tau T_q T'_q} & (-1)^{c-u-t'-T_q+m_t} a_{mc} b(mcqT'_q u)\langle\omega' t' \| V^{(2s)}_{\omega-\omega' u} \| \omega t\rangle \hat{u}\hat{t}'\hat{x} \\ & \times (2\tau+1)\begin{pmatrix} t & t' & x \\ m_t & -m'_t & m'_t - m_t \end{pmatrix}\begin{Bmatrix} t & t' & x \\ c & T_q & \tau \end{Bmatrix}\begin{Bmatrix} t & t' & u \\ c & T'_q & \tau \end{Bmatrix} \\ & \times \left(z^{m(c)} \times Z^{(q0)}_{T_q}(\nabla)\right)^{(x)}_{m_t - m'_t} \left(KK^\dagger(\omega t q\tau)\right)^{-1}_{T'_q T_q} \psi_{\omega t m_t}(z). \qquad (34)\end{aligned}$$

Comparison with the right-hand side of equation (28) shows that the coefficient of $\psi_{\omega t m_t}(z)$ in equation (34) may be identified with $g_{m'_t m_t}$. Now, using equation (24) and carrying out the sum over m gives

$$\begin{aligned} g^{(r)} = & \sum_u \langle\omega' t' \| V^{(2s)}_{\omega-\omega' u} \| \omega t\rangle (-1)^u \hat{u}\hat{r} \sum_{mcT_qT'_q} (-1)^{c-T_q} a_{mc} (z^{m(c)} \times Z^{(q0)}_{T_q}(\nabla))^{(r)} \\ & \times b(mcqT'_q u) \sum_\tau (2\tau+1) \begin{Bmatrix} t & t' & r \\ c & T_q & \tau \end{Bmatrix}\begin{Bmatrix} t & t' & u \\ c & T'_q & \tau \end{Bmatrix} \\ & \times (KK^\dagger(\omega t q\tau))^{-1}_{T'_q T_q}. \qquad (35)\end{aligned}$$

This expression for $g^{(r)}$ may now be inserted into equation (25) to obtain the matrix elements of $\Gamma(V)$. One need only calculate the matrix elements of the z and ∇ operators in equation (35) in the z-space. However, a further simplification may be made by decoupling z and ∇ in equation (35) which enables the r-sum in equation (25) to be carried out, giving finally

$$\begin{aligned} & ((\omega' t')p' T'_p T M_T | \Gamma(V^{(2s)}) | (\omega t) p T_p T M_T) \\ & \quad = (-1)^{T_p + T'_p + t'} \hat{T}_p \hat{T}'_p \hat{t}' \sum_u (-1)^u \hat{u} \langle\omega' t' \| V^{(2s)}_{\omega-\omega' u} \| \omega t\rangle \\ & \quad \times \sum_{mcT_qT'_q} a_{mc} b(mcT'_q u) d_{qT_q} \sum_y ((p'0)T'_p \| z^{m(c)} \| (p-q\ 0)y) \\ & \quad \times ((p0)T_p \| z^{q(T_q)} \| (p-q\ 0)y) \sum_\tau (-1)^\tau (2\tau+1) \\ & \quad \times \begin{Bmatrix} t & t' & u \\ c & T'_q & \tau \end{Bmatrix}\begin{Bmatrix} T & y & \tau \\ T_q & t & T_p \end{Bmatrix}\begin{Bmatrix} T & y & \tau \\ c & t' & T'_p \end{Bmatrix} (KK^\dagger(\omega t q\tau))^{-1}_{T'_q T_q}. \end{aligned}$$

(36)

Table 2. The coefficients $b(mcqT_q'u)$ in equation (32) for the O(5) representations $(2s)$ with $s = 0$ and $s = 2$.

					$s=0$			$s=2$		
ω'	m	c	q	T_q'	$u=0$	$u=1$	$u=2$	$u=0$	$u=1$	$u=2$
$\omega+2$	2	s	0	0	1	0	0	0	0	1
$\omega+1$	1	1	0	0	0	1	0	0	1	0
	2	s	1	1	0	$-\sqrt{2}$	0	0	$-\sqrt{5}$	$-\sqrt{3}$
ω	0	0	0	0	1	0	0	1	0	0
	1	1	1	1	$\sqrt{5}$	0	$-\sqrt{2}$	$-\sqrt{8}$	$-3/\sqrt{2}$	$-1/\sqrt{2}$
	2	s	2	0	$\sqrt{5}$	0	0	0	0	1
	2	s	2	2	0	0	$\sqrt{2}$	$\sqrt{20}$	$\sqrt{15}$	$\sqrt{7}$

In this expression, $d_{q\,T_q}$ is the normalization constant for the function $Z_{T_q}^{(p0)}$, taking the values $d = 1$ for $q = 0$ or 1 and $d = 1/\sqrt{2}$ or $-1/\sqrt{2}$ for $q = 2$ and $T_q = 2$ or 0 respectively. The range of values for m and c is given in table 1 and for τ, T_q and T_q' in appendix B. Recall that $q = m + \omega - \omega'$. The full set of matrix elements of $KK^\dagger$ required in equation (36) is given in appendix B. The values of u in the reduced matrix elements of $V^{(2s)}$ in this expression, for given (ωt) and $(\omega' t')$, are determined by (i) the $H_1 T$ structure of the representation $(2s)$ and (ii) the triangular condition on utt'. For each $\omega - \omega'$ the values of u which satisfy condition (i) are given in table 2. The number of allowed u values is equal to the multiplicity of $(\omega' t')$ in the O(5) reduction of the product $(2s) \times (\omega t)$. Only six reduced matrix elements of the $z^{m(c)}$ operators are required in equation (36). Since the z-space wavefunction $|(p0)T_p)$ belongs to the SU(3) representation $(p0)$ and the operator $z^{m(c)}$ belongs to $(m0)$, the matrix elements are given in terms of the well-known SU(3) Wigner coefficients

$$((p'0)T_p'\|z^{m(c)}\|(p0)T_p) = \delta_{p',m+p}\langle(p0)T_p;(m0)c\|(p'0)T_p'\rangle((p'0)|\!|\!|z^{(m0)}|\!|\!|(p0)). \tag{37}$$

We use the Wigner coefficients listed in table 2 of [12] together with the twice-reduced matrix elements $(p|\!|\!|z^{(00)}|\!|\!|p) = 1$, $(p+1|\!|\!|z^{(10)}|\!|\!|p) = (p+1)^{1/2}$ and $(p+2|\!|\!|z^{(20)}|\!|\!|p) = ((p+1)(p+2))^{1/2}$.

The computer algebra package REDUCE has been used to evaluate the formula (36) for the matrix elements of $\Gamma(V)$, for all states with even n and reduced isospin t =0, 1 and 2 for arbitrary j, in terms of the reduced matrix elements of $V^{(2s)}$. The results are given in appendix C. The reduced matrix elements of $V^{(2s)}$ are clearly independent of n and T and may therefore be treated as parameters in a discussion of the nT-dependence of an arbitrary interaction. However, their relation to the two-body matrix elements of the interaction is discussed in the next section.

To obtain the final shell-model matrix elements of V we must return to equations (17) and (18) and insert the K-matrix elements. Formulae for the $KK^\dagger$ matrices were given in [11, 12] for reduced isospin $t = 0, \frac{1}{2}$,1 and $\frac{3}{2}$, for the two-fold multiplicity in $t = 2$ and for small p. In appendix D we give formulae for the three-fold multiplicity in $t = 2$, which is then sufficient for all states up to seniority $v = 4$.

4. The reduced matrix elements of $V^{(2s)}_{\omega-\omega' u}$

The two-particle creation operators belong to O(5) representations (11) and (10) for even and odd J, respectively, and, in the notation of [12], are written

$$T^{(11)}_{11M_T}(JM) = \sqrt{\tfrac{1}{2}}(a^\dagger \times a^\dagger)^{J1}_{MM_T} \qquad T^{(10)}_{100}(JM) = \sqrt{\tfrac{1}{2}}(a^\dagger \times a^\dagger)^{J0}_{M0} \tag{38}$$

where the suffices denote H_1, T and M_T. Hence the general two-body interaction may be written

$$V = -\sqrt{3}\sum_{J \text{ even}} E_J(T^{(11)}_{11}(J)\dot{\times}T^{(11)}_{-11}(J))^{(0)} + \sum_{J \text{ odd}} E_J(T^{(10)}_{10}(J)\dot{\times}T^{(10)}_{-10}(J))^{(0)} \tag{39}$$

where the destruction operators are given by

$$T^{(1x)}_{-1T-M_T}(J-M) = (T^{(1x)}_{1TM_T}(JM))^\dagger(-1)^{M+M_T} = \sqrt{\tfrac{1}{2}}(a \times a)^{JT}_{-M-M_T} \tag{40}$$

and the notation $\dot{\times}$ means a dot product in J and a tensor product in T. With the help of the O(5) Wigner coefficients, we now break the interaction into its O(5) constituents

$$V = \sum_{ab} V^{(ab)}_{00} \tag{41}$$

where the sum runs over $(ab) = (22)$, (20), (11) and (00). In detail,

$$\begin{aligned} V^{(ab)}_{00} = -\sqrt{3}\sum_{J \text{ even}} & E_J\langle(11)11;(11)-11\|(ab)00\rangle T^{(ab)}_{00}(J \text{ even}) \\ & + \sum_{J \text{ odd}} E_J\langle(11)10;(11)-10\|(ab)00\rangle T^{(ab)}_{00}(J \text{ odd}) \end{aligned} \tag{42}$$

where the operators $T^{(ab)}_{00}$ are special cases of the O(5) tensor products

$$T^{(ab)}_{hu}(J \text{ even}) = \sum_{h_1t_1t_2}\langle(11)h_1t_1;(11)h-h_1t_2\|(ab)hu\rangle(T^{(11)}_{h_1t_1}(J)\dot{\times}T^{(11)}_{h-h_1t_2}(J))^{(u)} \tag{43}$$

$$T^{(ab)}_{hu}(J \text{ odd}) = \sum_{h_1t_1t_2}\langle(10)h_1t_1;(10)h-h_1t_2\|(ab)hu\rangle(T^{(10)}_{h_1t_1}(J)\dot{\times}T^{(10)}_{h-h_1t_2}(J))^{(u)}. \tag{44}$$

From equations (42)–(44) we have, not only the separate O(5) components $V^{(ab)}_{00}$ of the interaction, but also the definition of what might be called the 'extensions' $V^{(ab)}_{hu}$ of the interaction which we require in equation (36). The O(5) Wigner coefficients are given in table 3 and some simplification of equations (43) and (44) may be made by using commutation of the T-operators. (In particular this reduces the antisymmetric operator $V_{00}^{(11)}$ to the simple form given in equation (20).) In fact, we need matrix elements of the $V_{\omega-\omega' u}^{(ab)}$ between states of full seniority $n = v$.

For matrix elements diagonal in the seniority, the operator forms for $V_{0u}^{(ab)}$ are given in table 4 and equations (45)–(47)

$$V_{01}^{(22)} = \sqrt{\tfrac{2}{3}}(-E_0 + \Omega^{-1} \sum_{J \text{ even}} (2J+1)E_J)T^{(1)} + \sqrt{\tfrac{1}{3}} \sum_{J \text{ even}} (E_J + 2F_J^{e})(T_{11}^{(11)}(J)\dot{\times}T_{-11}^{(11)}(J))^{(1)} \tag{45}$$

$$V_{02}^{(20)} = \sqrt{\tfrac{2}{15}} \sum_{J \text{ even}} (-E_J + F_J^{e} - 3F_J^{o})(T_{11}^{(11)}(J)\dot{\times}T_{-11}^{(11)}(J))^{(2)} \tag{46}$$

$$V_{02}^{(22)} = -\sqrt{\tfrac{1}{3}} \sum_{J \text{ even}} (E_J + 2F_J^{e})(T_{11}^{(11)}(J)\dot{\times}T_{-11}^{(11)}(J))^{(2)} \tag{47}$$

where the F_J are defined

$$F_J = \sum_{J'} (2J'+1)E_{J'} \begin{Bmatrix} j & j & J' \\ j & j & J \end{Bmatrix} \tag{48}$$

and $T^{(1)}$ is the isospin operator. The upper index 'e' or 'o' on F_J means that the sum over J' is restricted to even or odd J' respectively. When $v = 0$ or 1, the operators in the last six columns of table 4 and in equations (45)–(47) give no contribution. For $v = 2$, the matrix elements of these operators are simply

$$\langle j^2 \tilde{J} t \| (T_{1x}^{(1x)}(J)\dot{\times}T_{-1x}^{(1x)}(J))^{(u)} \| j^2 \tilde{J} t \rangle = (-1)^x \delta_{\tilde{J}J} \delta_{xt} \hat{u}/\hat{x} \tag{49}$$

while for seniority $v = 4$, they can be given only in terms of the two-body fractional parentage coefficients which define the state of seniority four,

$$\begin{aligned} \langle j^4 \alpha' \tilde{J} t' \| & (T_{1x}^{(1x)}(J)\dot{\times}T_{-1x}^{(1x)}(J))^{(u)} \| j^4 \alpha \tilde{J} t \rangle \\ &= \sum_{J'x'} (-1)^{u+x'+t'} 6\hat{u}\hat{t} \begin{Bmatrix} t' & t & u \\ x & x & x' \end{Bmatrix} \\ &\quad \times \langle j^4 \alpha' \tilde{J} t' \{\!| j^2 J' x', j^2 J x \rangle \langle j^4 \alpha \tilde{J} t \{\!| j^2 J' x', j^2 J x \rangle . \end{aligned} \tag{50}$$

For a change of two units in seniority, $v' = v - 2$, equation (36) needs reduced matrix elements of $V_{-1u}^{(2s)}$. Angular momentum conservation prevents any such term for $v = 2$, $v' = 0$. For $v = 4$, $v' = 2$ the required matrix elements are given in table 5 in terms of the quantities

$$f_{x'x} = \sum_{JJ'} (-1)^{\tilde{J}} \hat{J} \hat{J}' E_J \begin{Bmatrix} J & \tilde{J} & J' \\ j & j & j \end{Bmatrix} \langle j^4 \alpha \tilde{J} t \{| j^2 J' x', j^2 J x \rangle \tag{51}$$

where the sums are restricted by the conditions that both $x + J$ and $x' + J'$ should be odd. The isospin labels x, x' may be 0 or 1.

Table 3. The O(5) Wigner coefficients needed in the analysis of the interaction.

		(10) × (10)						(11) × (11)										
	h	0			−1		−2	0						−1				−2
	$h_1\, t_1$	−1 0	0 1	1 0	−1 0	0 1	−1 0	−1 1	0 0	0 0	0 1	0 1	1 1	−1 1	−1 1	0 0	0 1	−1 1
	$h_2\, t_2$	1 0	0 1	−1 0	0 1	−1 0	−1 0	1 1	0 0	0 1	0 0	0 1	−1 1	0 0	0 1	−1 1	−1 1	−1 1
(0 0)	$u=0$	$\sqrt{\frac{1}{5}}$	$-\sqrt{\frac{3}{5}}$	$\sqrt{\frac{1}{5}}$				$\sqrt{\frac{3}{10}}$	$-\sqrt{\frac{1}{10}}$			$\sqrt{\frac{3}{10}}$	$\sqrt{\frac{3}{10}}$					
(2 0)	$u=0$	$-\sqrt{\frac{3}{10}}$	$-\sqrt{\frac{2}{5}}$	$-\sqrt{\frac{3}{10}}$			1	$\sqrt{\frac{1}{30}}$	$-\sqrt{\frac{2}{5}}$			$-\sqrt{\frac{8}{15}}$	$\sqrt{\frac{1}{30}}$					1
	$u=1$				$\sqrt{\frac{1}{2}}$	$\sqrt{\frac{1}{2}}$								$\sqrt{\frac{1}{6}}$	$-\sqrt{\frac{1}{3}}$	$\sqrt{\frac{1}{6}}$	$\sqrt{\frac{1}{3}}$	
	$u=2$		1					$\sqrt{\frac{1}{3}}$				$-\sqrt{\frac{1}{3}}$	$\sqrt{\frac{1}{3}}$					
(2 2)	$u=0$							$\sqrt{\frac{1}{6}}$	$\sqrt{\frac{1}{2}}$			$-\sqrt{\frac{1}{6}}$	$\sqrt{\frac{1}{6}}$					
	$u=1$							$\sqrt{\frac{1}{6}}$		$\sqrt{\frac{1}{3}}$	$\sqrt{\frac{1}{3}}$		$-\sqrt{\frac{1}{6}}$	$\sqrt{\frac{1}{3}}$	$\sqrt{\frac{1}{6}}$	$\sqrt{\frac{1}{3}}$	$-\sqrt{\frac{1}{6}}$	
	$u=2$							$\sqrt{\frac{1}{6}}$				$\sqrt{\frac{2}{3}}$	$\sqrt{\frac{1}{6}}$	$\sqrt{\frac{1}{2}}$	$\sqrt{\frac{1}{2}}$		$\sqrt{\frac{1}{2}}$	1
(1 1)	$u=0$							$\sqrt{\frac{1}{2}}$					$-\sqrt{\frac{1}{2}}$					

Table 4. The coefficients in the expressions for $V_{00}^{(a\,b)}$ in states of full seniority $n=v$. The sums over J are restricted to even or odd J for $x=1$ or 0 respectively.

	E_0	$\sum_e (2J+1)E_J$	$\sum_o (2J+1)E_J$	$\sum_J E_J(T^{(1\,x)}_{1\,x}(J)\dot{\times}T^{(1\,x)}_{-1\,x}(J))^{(0)}$		$\sum_J F^e_J(T^{(1\,x)}_{1\,x}(J)\dot{\times}T^{(1\,x)}_{-1\,x}(J))^{(0)}$		$\sum_J F^o_J(T^{(1\,x)}_{1\,x}(J)\dot{\times}T^{(1\,x)}_{-1\,x}(J))^{(0)}$	
				$x=1$	$x=0$	$x=1$	$x=0$	$x=1$	$x=0$
$V_{00}^{(0\,0)}$	$3(\Omega-v)/10$	$3(6\Omega-v)/20\Omega$	$(4\Omega+v)/20\Omega$	$-3\sqrt{3}/5$	$2/5$	$3\sqrt{3}/5$	$-3/5$	$\sqrt{3}/5$	$-3/5$
$V_{00}^{(2\,0)}$	$(\Omega-v)/5$	$(\Omega-v)/10\Omega$	$3(\Omega-v)/10\Omega$	$-1/5\sqrt{3}$	$3/5$	$1/5\sqrt{3}$	$3/5$	$-\sqrt{3}/5$	$3/5$
$V_{00}^{(2\,2)}$	$(v-\Omega)/2$	$(\Omega-v)/2\Omega$	0	$-1/\sqrt{3}$	0	$-2/\sqrt{3}$	0	0	0

Table 4. The reduced matrix elements $\langle j^2\,\tilde{J}t'||V^{(2s)}_{-1u}||j^4\,\alpha\,\tilde{J}t\rangle$ in terms of the $f_{x'x}$ given in equation (51).

t'	t	$s=0,\ u=1$	$s=2,\ u=1$	$s=2,\ u=2$
0	1	$-2\sqrt{\frac{3}{5}}(\sqrt{2}f_{11}-f_{01}+3f_{10})$	$2\sqrt{3}(f_{11}+\sqrt{2}f_{01})$	—
	2	—	—	$-2\sqrt{15}f_{11}$
1	0	$2\sqrt{\frac{3}{5}}(f_{11}+\sqrt{3}f_{00})$	0	—
	1	$-2\sqrt{\frac{2}{5}}(\sqrt{2}f_{11}-f_{01}+3f_{10})$	$-\sqrt{2}(f_{11}+\sqrt{2}f_{01})$	$\sqrt{10}(f_{11}+\sqrt{2}f_{01})$
	2	0	$\sqrt{30}f_{11}$	$-\sqrt{30}f_{11}$

For a change of four units in seniority, only $V^{(2s)}_{-2u}$ contributes and with $v=4$, $v'=0$ the required matrix elements are

$$\begin{aligned}\langle 0||V^{(20)}_{-20}||j^4\alpha 00\rangle &\\ &= \sqrt{\tfrac{3}{5}}\Big[-\sum_{J\ \text{even}} \hat{J}E_J\langle j^4\alpha 00\{|j^2J1,\ j^2J1\rangle \\ &\quad + \sqrt{3}\sum_{J\ \text{odd}} \hat{J}E_J\langle j^4\alpha 0\{|j^2J0, j^2J0\rangle\Big]\end{aligned} \tag{52}$$

$$\langle 0||V^{(22)}_{-22}||j^4\alpha 02\rangle = -\sqrt{15}\sum_{J\ \text{even}} \hat{J}E_J\langle j^4\alpha 02\{|j^2J1, j^2J1\rangle. \tag{53}$$

5. Discussion

Using vector coherent state theory, we have derived a general formula (36) for the matrix elements of each irreducible O(5) component of a general isospin invariant two-body interaction in the orthonormal shell-model basis introduced in an earlier paper [11]. Analytic results are given for even n and $t=0$, 1 and 2 which is sufficient for seniority $v\leqslant 4$ in terms of a minimum number of reduced matrix elements which define the interaction and the dependence on the conserved quantum numbers n and T is clearly shown. Although the calculation of reduced matrix elements in section 4 is specific to a single j-shell, the results in earlier sections, including appendix C, are also valid for a set of j-shells with $\Omega=\sum_j(j+\frac{1}{2})$.

As an illustration consider the case $j=7/2$, $v=2$, $t=1$, $n=6$, $J=2$, $T=1$. Here $p=2$, so that $T_p=2$ or 0 and there are therefore two independent states. This is because, in the U(8)⊃Sp(8)⊃O(3) classification scheme [13], the (1100) representation of Sp(8) occurs twice in the representation [2211] of U(8). The matrix elements of Γ(V) are given below in the notation on the right-hand side of equations (17) and (18):

$$(2|\Gamma(V)|2) = \langle 31\|V_{00}^{(00)}\|31\rangle + \frac{1}{3}\langle 31\|V_{00}^{(11)}\|31\rangle + \frac{17}{72}\langle 31\|V_{00}^{(20)}\|31\rangle + \frac{29}{72\sqrt{2}}\langle 31\|V_{02}^{(20)}\|31\rangle + \frac{5}{126}\langle 31\|V_{00}^{(22)}\|31\rangle - \frac{9}{7\sqrt{3}}\langle 31\|V_{01}^{(22)}\|31\rangle + \frac{2}{63\sqrt{5}}\langle 31\|V_{02}^{(22)}\|31\rangle \tag{54}$$

$$(0|\Gamma(V)|2) = \frac{\sqrt{5}}{63}\langle 31\|V_{00}^{(20)}\|31\rangle - \frac{11\sqrt{5}}{63\sqrt{2}}\langle 31\|V_{02}^{(20)}\|31\rangle + \frac{2\sqrt{5}}{45}\langle 31\|V_{00}^{(22)}\|31\rangle - \frac{\sqrt{15}}{20}\langle 31\|V_{01}^{(22)}\|31\rangle + \frac{11}{36}\langle 31\|V_{02}^{(22)}\|31\rangle \tag{55}$$

$$(2|\Gamma(V)|0) = \frac{\sqrt{5}}{36}\langle 31\|V_{00}^{(20)}\|31\rangle - \frac{11\sqrt{5}}{36\sqrt{2}}\langle 31\|V_{02}^{(20)}\|31\rangle + \frac{11\sqrt{5}}{315}\langle 31\|V_{00}^{(22)}\|31\rangle - \frac{2\sqrt{15}}{35}\langle 31\|V_{01}^{(22)}\|31\rangle + \frac{32}{63}\langle 31\|V_{02}^{(22)}\|31\rangle \tag{56}$$

$$(0|\Gamma(V)|0) = \langle 31\|V_{00}^{(00)}\|31\rangle + \frac{1}{3}\langle 31\|V_{00}^{(11)}\|31\rangle + \frac{17}{63}\langle 31\|V_{00}^{(20)}\|31\rangle + \frac{\sqrt{2}}{63}\langle 31\|V_{02}^{(20)}\|31\rangle - \frac{47}{45}\langle 31\|V_{00}^{(22)}\|31\rangle + \frac{3\sqrt{3}}{10}\langle 31\|V_{01}^{(22)}\|31\rangle - \frac{1}{18\sqrt{5}}\langle 31\|V_{02}^{(22)}\|31\rangle \tag{57}$$

in terms of the reduced matrix elements $\langle \omega t\|V_{0u}^{(ab)}\|\omega t\rangle$. These results have been obtained by using tables C1 and C2 with $\omega = \omega' = 3$ and $t = t' = 1$. The last two equations require exploitation of the symmetry mentioned in appendix C. This is achieved by inserting $T = -2$ in the listed formulae rather than $T = 1$. The reduced matrix elements in these equations can be evaluated as linear combinations of the E_J by use of table 4 and equations (45)–(49). Using values for the E_J derived [14] from the experimental levels in ^{42}Sc we find, in MeV, $\langle 31\|V_{00}^{(00)}\|31\rangle = -28.76$, $\langle 31\|V_{00}^{(11)}\|31\rangle = 40.32$, $\langle 31\|V_{00}^{(20)}\|31\rangle = -13.40$, $\langle 31\|V_{02}^{(20)}\|31\rangle = -3.72$, $\langle 31\|V_{00}^{(22)}\|31\rangle = 0.34$, $\langle 31\|V_{01}^{(22)}\|31\rangle = 0.29$, $\langle 31\|V_{02}^{(22)}\|31\rangle = 0.08$. This leads to the unsymmetric matrix

$$\Gamma(V) = \begin{array}{c} (T_p = 0) \\ (T_p = 2) \end{array} \begin{pmatrix} -19.23 & 0.55 \\ 0.96 & -19.75 \end{pmatrix} \tag{58}$$

which has eigenvalues -18.72 MeV, -20.26 MeV. The final shell-model matrix is now obtained by means of the transformation $K^{-1}\Gamma(V)K$. The matrix $KK^\dagger$ for this case is available from [11], and K may then be calculated by either of the prescriptions in equations (12) and (13). In particular equation (13) gives

$$K = \begin{array}{c} (T_p = 0) \\ (T_p = 2) \end{array} \begin{pmatrix} 3.02574 & 0.10740 \\ 0.10740 & 3.91431 \end{pmatrix} \tag{59}$$

and the resulting matrix is

$$V = \begin{array}{c} (T_p = 0) \\ (T_p = 2) \end{array} \begin{pmatrix} -19.24 & 0.73 \\ 0.73 & -19.74 \end{pmatrix}. \tag{60}$$

This is symmetric as expected and has the same eigenvalues as $\Gamma(V)$. It is clear that there is considerable mixing of T_p in the eigenstates, confirming the conclusion of earlier work [5]. A curious feature of (60) is that its eigenvectors are actually independent of the interaction used. This is a special feature of the $j = 7/2$ shell and does not occur in higher shells.

The physical significance of this illustration is that it refers to the nucleus ^{46}Ti and typifies the description of the lowest 2^+ states in spherical even–even nuclei. It was shown in [5] that the mixing of T_p leads to a lowest 2^+ state very close to having 'full symmetry' with respect to the group U(6) in a boson picture of nucleon pairs [15, 16]. The orthogonal partner has 'mixed symmetry' in this sense and there has been considerable interest [17, 18] in identifying such states and measuring the purity of the U(6) symmetry. The existence of the general shell-model results in this paper will enable us to study the boson mapping in detail and, in particular, to investigate the way that it depends on the conserved quantum numbers n and T.

Appendix A. The two orthonormal shell-model bases

We first note that the bases (17) and (18) have the same traces

$$\sum_i \langle i|V|i\rangle = \sum_{T_p} \langle T_p|V|T_p\rangle = \sum_{\tilde{T}_p} (\tilde{T}_p|\Gamma(V)|\tilde{T}_p) \tag{A.1}$$

and

$$\sum_{ij} \langle i|V|j\rangle\langle j|V|i\rangle = \sum_{T'_pT_p} \langle T'_p|V|T_p\rangle\langle T_p|V|T'_p\rangle$$
$$= \sum_{\tilde{T}'_p\tilde{T}_p} (\tilde{T}'_p|\Gamma(V)|\tilde{T}_p)(\tilde{T}_p|\Gamma(V)|\tilde{T}'_p). \tag{A.2}$$

These relations are useful in checking since the z-space matrix elements on the right-hand side do not involve the K matrices for the many-nucleon states.

Table A1. The i-basis expansion coefficients of equation (17) for shell-model matrix elements $\langle i'|V|i\rangle$ in terms of z-space matrix elements $(\tilde{T}'_p|\Gamma(V)|\tilde{T}_p)$ for the case $j = 11/2$, $n = 6$, $v = 2$, $t = 1$, $T = 1$.

	$(0\|\Gamma(V)\|0)$	$(0\|\Gamma(V)\|2)$	$(2\|\Gamma(V)\|0)$	$(2\|\Gamma(V)\|2)$
$\langle 0\|V\|0\rangle$	0.99351	−0.08026	−0.08026	0.00648
$\langle 0\|V\|2\rangle$	0.09345	1.15681	−0.00755	−0.09345
$\langle 2\|V\|0\rangle$	0.06893	−0.00557	0.85327	−0.06893
$\langle 2\|V\|2\rangle$	0.00648	0.08026	0.08026	0.99351

It has been pointed out [11] that the matrices U are nearly diagonal in many realistic examples, with each state i being dominated by a particular T_p. If as an approximation U is taken to be diagonal, it must be the unit matrix and the two bases (14) and (15) are identical. Both equations (17) and (18) then reduce to

$$\langle T_p'|V|T_p\rangle \approx \sqrt{\lambda/\lambda'}(T_p'|\Gamma(V)|T_p). \tag{A.3}$$

The lack of symmetry on the right-hand side of this equation is due to the lack of unitarity of Γ. For a Hermitian V this may be exploited since also

$$\langle T_p'|V|T_p\rangle^* \approx \sqrt{\lambda'/\lambda}(T_p|\Gamma(V)|T_p') \tag{A.4}$$

which leads to

$$\langle T_p'|V|T_p\rangle \approx \left\{(T_p'|\Gamma(V)|T_p)(T_p|\Gamma(V)|T_p')^*\right\}^{1/2}. \tag{A.5}$$

In this approximation, the λ have been eliminated so that no knowledge of the many-body K matrices is required. It is, however, necessary to know both the matrix elements on the right-hand side of the equation and, in many cases, one of these is more complicated to calculate than the other, see section 3.

It is worth commenting that the exact eigenvalues of $\Gamma(V)$ are the same as those of V. Hence the introduction of the K-matrices in equations (17) or (18) is not strictly necessary in finding the eigenvalues. However, this requires the *full* matrix of $\Gamma(V)$, which is not symmetric. With the help of equations (17) or (18) it is necessary to calculate only one half of the $\Gamma(V)$ matrix, taking advantage of the symmetry of the V matrix.

To compare the merits of the two bases (17) and (18) it is clearly necessary to go beyond the simple approximation of assuming U to be diagonal. We therefore investigate some examples.

The case $j = 11/2$, $n = 6$, $v = 2$, $t = 1$, $T = 1$.

In this example, the number of pairs is $p = (n - v)/2 = 2$ so that $T_p = 2$ or 0 and $KK^\dagger$ is a two-by-two matrix. From equation (25) of [11] this matrix is easily constructed and diagonalized to give $\lambda_0 = 26.1065$, $\lambda_2 = 35.3935$, with

$$U^{-1} = U^\dagger = \begin{array}{c} \\ (T_p = 0) \\ (T_p = 2) \end{array} \begin{array}{c} \begin{array}{cc} (i=0) & (i=2) \end{array} \\ \begin{pmatrix} 0.99675 & 0.08052 \\ -0.08052 & 0.99675 \end{pmatrix} \end{array}. \tag{A.6}$$

With these numbers, the coefficients of each of the matrix elements on the right-hand side in equations (17) and (18) are soon calculated and the results are given in tables A1 and A2.

Table A2. As table A1 for the orthonormal T_p-basis of equation (18)

	$(0\|\Gamma(V)\|0)$	$(0\|\Gamma(V)\|2)$	$(2\|\Gamma(V)\|0)$	$(2\|\Gamma(V)\|2)$
$\langle 0\|V\|0\rangle$	1.00015	0.01318	−0.01134	−0.00015
$\langle 0\|V\|2\rangle$	0.01318	1.16220	−0.00015	−0.01318
$\langle 2\|V\|0\rangle$	−0.01134	−0.00015	0.86068	0.01134
$\langle 2\|V\|2\rangle$	−0.00015	−0.01318	0.01134	1.00015

It is clear from the tables that the off-diagonal coefficients in table A2 are much smaller than those in table 1. There is also further cancellation in table A2 because, although the matrix of $\Gamma(V)$ is not symmetric, even for Hermitian V, it is approximately symmetric in practice. Hence there will, for example be approximate cancellation between $(0|\Gamma(V)|2)$ and $(2|\Gamma(V)|0)$ in the expression for $\langle 0|V|0\rangle$. Notice that the diagonal coefficients for $T_p' = T_p$ in table 2 are very close to unity. In addition the product of the diagonal coefficients for $\langle 0|V|2\rangle$ and $\langle 2|V|0\rangle$ is very close to unity (1.00028) and so justifies the approximation (A.5). A similar calculation for the case $j = 15/2$, $n = 12$, $v = 4$, $t = 2$, $T = 2$ has off-diagonal coefficients in the i-basis of table A1 up to 0.51 while the greatest such number in the T_p-basis of table A2 is 0.10. We therefore conclude that the approximation (A.5) is good in the T_p-basis but would be much less accurate in the i-basis.

In the basis (5), the rows and columns of $KK^\dagger$, for particular p and T, are labelled by $T_p = p, (p-2), \ldots, 1$ or 0 subject to the triangle constraint $|T - t| \leqslant T_p \leqslant T + t$. These conditions determine the size N_p of the matrix. However, for values of p close to its maximum of $2j + 1 - v$, the Pauli principle may restrict the number of shell-model states to some number $\nu_p < N_p$. From equation (14) it may be seen that the 'missing' states, which vanish identically, correspond to the zero eigenvalues of $KK^\dagger$ and hence that the physical states are restricted to $i = 1, 2, \ldots, \nu_p$ with $\lambda_i > 0$. Although the K matrix is then singular we may still use the expression $(K^\dagger)^{-1}_{T_p i} = U^{-1}_{T_p i}/\sqrt{\lambda_i}$ from equation (12) for the restricted set of states i in equation (14). In this exceptional situation, the basis (15) loses its orthonormality, and its linear independence, because of the vanishing norm of the missing states. The zero eigenvalues would be omitted in using equation (13) to construct the $(K^\dagger)^{-1}$ matrix elements in equation (15). Since, in practice, each i-state is found to be dominated by a particular T_p, one could omit those shell-model states labelled by the T_p which dominate the excluded i and apply a Gram–Schmidt orthonormalization to the remaining T_p.

Appendix B. The $KK^\dagger$ matrices in equation (36)

For the $KK^\dagger$ matrices in equation (36) we are concerned only with small numbers of pairs $q = 0$, 1, and 2 and the $KK^\dagger$ were given in [12]. (Note that the $KK^\dagger$ matrices in equation (17) and (18) may have a large number p of pairs.) The results are:

For $q = 0$, trivially $KK^\dagger = 1$ with $T_q = T_q' = 0$ and $\tau = t$.

For $q = 1$, $KK^\dagger$ is one-dimensional with $T_q = T_q' = 1$ and takes values $(\omega + t + 1)$, $(\omega + 1)$ and $(\omega - t)$ for $\tau = t - 1$, t and $t + 1$ respectively.

For $q = 2$, $KK^\dagger$ is again one-dimensional for $\tau = t - 2$, $t - 1$, $t + 1$, and $t + 2$ with values $(\omega + t)(\omega + t + 1)$, $(\omega + 1)(\omega + t + 1)$, $(\omega + 1)(\omega - t)$, and $(\omega - t)(\omega - t - 1)$ respectively. For $\tau = t$, there is, except for very small $\omega = t$, a two-dimensional matrix for $KK^\dagger$ with $T_q = 0$ and 2 which may be inverted to give

$$(KK^\dagger)^{-1}_{00} = \left\{(\omega + 1)^2 - t(t+1)/3\right\}/D$$

$$(KK^\dagger)^{-1}_{02} = (KK^\dagger)^{-1}_{20} = -\left\{t(t+1)(2t-1)(2t+3)/18\right\}^{1/2}/D$$

$$(KK^\dagger)^{-1}_{22} = \left\{\omega(\omega + \tfrac{1}{2}) - 2t(t+1)/3\right\}/D$$

where $D = (\omega + 1)(\omega + \frac{1}{2})(\omega - t)(\omega + t + 1)$.

Appendix C. The z-space matrix elements of $\Gamma(V)$ from equation (36)

The shell-model matrix elements of the component $V^{(11)}$ of the interaction are given by equation (20). For the invariant component $V^{(00)}$, the shell-model matrix elements are equal to the reduced matrix element, given in table 4 and equations (49) and (50) for $v = 2$ and 4. For the remaining components $V^{(2s)}$, table C1 ($s = 0$) and table C2 ($s = 2$) give the coefficients of each reduced matrix element $\langle\omega' t' \| V^{(2s)}_{\omega-\omega' u} \| \omega t\rangle$ in the expression (36) for the matrix element elements $((\omega' t')p' T'_p T | \Gamma(V^{(2s)}) | (\omega t) p T_p T)$, as functions of ω, p and T for each relevant T_p and T'_p. They must be multiplied by the appropriate reduced matrix element (see section 4) and summed over the different values of u given in the tables. Finally, to obtain the shell-model matrix element, the KK^{-1} factor from equations (17) and (18) must be included.

We recall that the symbols in the tables are related to physical quantities by $\omega = j + 1/2 - v/2$, $p = (n - v)/2$, $p' = p + \omega' - \omega$. The tables include all matrix elements needed for even n and $v \leqslant 4$ but they have more general validity. In each table, the entries are ordered first by the value of $\omega' - \omega = 2,1$ or 0, then by $t't$ and finally by T'_p and T_p. It is sufficient to consider $\omega' \geqslant \omega$ since the final matrix of V is symmetric, although $\Gamma(V)$ is not. For $\omega' = \omega$ it is similarly sufficient to restrict entries to $t' \leqslant t$. The range of the values of T_p and T'_p is governed by the rule that $T'_p - T_p + \omega' - \omega$ is even, together with the usual triangle rules on tT_pT and $t'T'_pT$. A few matrix elements of $\Gamma(V^{(20)})$ which satisfy these conditions nevertheless vanish and are omitted from the tables. They are $T'_p = T \mp 1$, $T_p = T \mp 2$ for $\omega' = \omega + 1$, $t' = 1$, $t = 2$ and $T'_p = T \pm 2$, $T_p = T \mp 2$ for $\omega' = \omega$, $t' = t = 2$. Entries for even and odd $T - T_p$ are grouped separately since the parities of $T - T_p$ and $T - p = T - n/2 + v/2$ are the same. The low states of even–even nuclei have $T - n/2$ even which implies $T - T_p$ even for $v = 0$ and 4 but $T - T_p$ odd for $v = 2$.

The number of entries in the table has been reduced by about 40% by taking advantage of an interesting symmetry which was noticed after the calculations were completed. The entries with a $\pm$ in the $T_pT'_p$ column must therefore be interpreted in the following way. The formulae as given apply only for the upper sign. To obtain the result for the lower sign, the substitution $T \to -(T+1)$ must be made in the formulae and the result must be multiplied by an overall sign of $(-1)^{t-t'}$. Each negative factor under a square root sign brings a factor $\sqrt{-1}$ so that, for example, $\sqrt{T(T+1)} \to -\sqrt{T(T+1)}$ whereas $T(T+1) \to T(T+1)$. Although we do not yet fully understand the origin of this symmetry, it is probably related to the concept of negative angular momentum, see [19, 20]. We stress that we have not assumed its validity but simply used it in tabulating separate results more concisely. The printed expressions in tables C1 and C2 have all been checked by comparison with direct numerical computation for particular values of ω, p and T.

Appendix D. The $KK^\dagger$ matrices

The one-dimensional cases $(\omega 0)$ and $(\omega 1)$ with $p - T$ even are given in equations (A5) of [12] and (24) of [11] respectively. The two-dimensional cases $(\omega 1)$ and $(\omega 2)$ both with $p - T$ odd are given in [11], equations (25) and (27) respectively. New analytic formulae for the case $(\omega 2)$ with $p - T$ even, which in general is three-

Table C1. (continued)

t'	t	T'_p	T_p	u	$((\omega'\,t')\,p'\,T'_p\,T\Vert\Gamma(V^{(20)})\Vert(\omega\,t)\,p\,T_p\,T)$
		$(\omega'=\omega)$			
2	2	$T\pm2$	$T\pm2$	0	$2\{5(p-\omega)^2(2\omega^2+4\omega+3)+(2\omega^4+10\omega^3-3\omega^2-57\omega-51)-5(2\omega^2+4\omega-3)T(T+1)\}/D$
					$+2\{(2T+1)[5(p-\omega)^2\omega(\omega+2)-5(\omega+2)\omega T(T+1)+(\omega^4+5\omega^3+6\omega^2-6\omega-3)]\}/D$
				2	$-\sqrt{10/7}\,\{(p-\omega)^2(10\omega^2+5\omega-12)+(\omega-4)(2\omega+9)T(T+1)-(10\omega^4+35\omega^3-15\omega^2-129\omega-96)\}/D$
					$-\sqrt{10/7}\,(2T+1)[(p-\omega)^2(2\omega^2+\omega+6)-(2\omega^2+\omega+6)T(T+1)-(2\omega^4+7\omega^3-9\omega^2-33\omega-12)]/D$
					$D=6(2\omega+1)(\omega+3)(\omega+1)(\omega-2)(2T+3)$
		T	$T\pm2$	0	$-5\sqrt{14}\,F$
				2	$(2\omega^2+\omega-8)\sqrt{5}\,F$
					$F=\sqrt{(p+T+3)(p-T)(2T-1)(T+2)T/21(2T+1)}\,(p-2\omega+T)/(2\omega+1)(\omega+3)(\omega+1)(\omega-2)(2T+3)$
		$T\pm2$	T	0	$-5\sqrt{14}\,F$
				2	$(2\omega^2+\omega-8)\sqrt{5}\,F$
					$F=\sqrt{(2T-1)(T+2)T(p+T+3)(p-T)/21(2T+1)}\,(p-2\omega-T-3)/(2\omega+1)(\omega+3)(\omega+1)(\omega-2)(2T+3)$
		T	T	0	$\{5(p-\omega)^2[4(\omega^2+2\omega-2)T(T+1)-3(\omega^2+2\omega-6)]-3(\omega^4+5\omega^3+36\omega^2+84\omega+42)\}/D$
					$+T(T+1)[-20(\omega^2+2\omega-2)T(T+1)+(4\omega^4+20\omega^3+79\omega^2+126\omega-42)]/D$
				2	$\sqrt{5/14}\,\{(p-\omega)^2[4(2\omega^2+\omega-22)T(T+1)-3(10\omega^2+5\omega-54)]+3(10\omega^4+35\omega^3-3\omega^2-102\omega-60)\}/D$
					$-\sqrt{5/14}\,T(T+1)[4(2\omega^2+\omega-22)T(T+1)+(8\omega^4+28\omega^3-10\omega^2-63\omega+114)]/D$
					$D=3(2\omega+1)(\omega+3)(\omega+1)(\omega-2)(2T+3)(2T-1)$
		$T\pm1$	$T\pm1$	0	$45[(p-\omega)^2+T(T+1)-(\omega^2+3\omega+2)]/D+(2T+1)[5(p-\omega)^2(2\omega^2+4\omega-3)-5(2\omega^2+4\omega-3)T(T+1)+(2\omega^4+10\omega^3+27\omega^2+33\omega-6)]/D$
				2	$-9\sqrt{5/14}\,(2\omega^2+\omega-8)[(p-\omega)^2+T(T+1)-(\omega+2)(\omega+1)]/D$
					$-\sqrt{5/14}\,(2T+1)[-(p-\omega)^2(2\omega+9)(\omega-4)+(2\omega+9)(\omega-4)T(T+1)+(2\omega^4+7\omega^3+3\omega^2-6\omega+24)]/D$
					$D=6(2\omega+1)(\omega+3)(\omega+1)(\omega-2)(2T+1)$
		$T\mp1$	$T\pm1$	0	$-5\sqrt{14}\,F$
				2	$(2\omega^2+\omega-8)\sqrt{5}\,F$
					$F=\sqrt{(p+T+2)(p-T+1)(T+2)(T-1)/14}\,(p-2\omega+T-1)/(2\omega+1)(\omega+3)(\omega+1)(\omega-2)(2T+1)$

Table C1. Coefficients of the reduced matrix elements $\langle \omega'\, t' \| \mathbf{V}^{(2\,0)}_{\omega-\omega'\,u} \| \omega\, t \rangle$ in expression (36).

t'	t	T'_p	T_p	u	$((\omega'\,t')\,p'\,T'_p\,T \Vert \Gamma(V^{(20)}) \Vert (\omega\,t)\,p\,T_p\,T)$
		$(\omega' = \omega + 2)$			
0	0	T	T	0	$\sqrt{5(p+T+3)(p-T+2)/6}$
		$(\omega' = \omega + 1)$			
0	1	T	$T \pm 1$	1	$-\sqrt{5(T+1)(p-T+1)/(2T+1)}\,(p-\omega+T)/3(\omega+2)$
1	0	$T \pm 1$	T	1	$\sqrt{5(T+1)(p+T+3)/3(2T+1)}\,(p-\omega-T)/\omega$
1	1	T	$T \pm 1$	1	$-\sqrt{5T(p-T+1)/6(2T+1)}\,(p-\omega+T+1)/(\omega+1)$
		$T \pm 1$	T		$-\sqrt{5T(p+T+3)/6(2T+1)}\,(p-\omega-T-1)/(\omega+1)$
1	2	$T \pm 1$	$T \pm 2$	1	$-\sqrt{(p-T)(T+2)/(2T+3)}\,(p+T-\omega)/(\omega+3)$
		$T \pm 1$	T		$\sqrt{(p+T+3)(2T-1)T/6(2T+3)(2T+1)}\,(p-T-\omega-3)/(\omega+3)$
		T	$T \pm 1$		$-\sqrt{(p-T+1)(T+2)/2(2T+1)}\,(p+T-\omega-1)/(\omega+3)$
		$(\omega' = \omega)$			
0	0	T	T	0	$\left[5(p-\omega)^2-5T(T+1)+\omega(\omega+3)\right]/3\omega(2\omega+1)$
0	2	T	$T \pm 2$	2	$\sqrt{(T+1)(T+2)(p+T+3)(p-T)/6(2T+1)(2T+3)}\,(p-2\omega+T-3)/(\omega+3)(\omega+2)$
		T	T		$\sqrt{T(T+1)/(2T-1)(2T+3)}\,[(2p+3)(\omega+2)-(p+T+1)(p-T)]/3(\omega+3)(\omega+2)$
1	1	$T \pm 1$	$T \pm 1$	0	$\sqrt{2}\left\{15\left[(p-\omega)^2-(\omega+1)(\omega+2)+T(T+1)\right]+3(2T+1)\left[5(p-\omega)^2(2\omega^2+4\omega+1)+(2\omega^2+4\omega-1)(\omega+2)(\omega+1)-5(2\omega^2+4\omega+1)T(T+1)\right]\right\}/D$
				2	$\left\{-3(6\omega^2+3\omega-8)\left[(p-\omega)^2-(\omega+1)(\omega+2)+T(T+1)\right]-3(2T+1)\left[(p-\omega)^2(2\omega^2+\omega+4)-(2\omega^2+\omega+4)T(T+1)-(2\omega^2+\omega-4)(\omega+2)(\omega+1)\right]\right\}/D$
					$D=18\sqrt{2}\,(2\omega+1)(\omega+2)(\omega+1)(\omega-1)(2T+1)$
		$T \mp 1$	$T \pm 1$	0	$-5\sqrt{2}\,F$
				2	$(6\omega^2+3\omega-8)F$
					$F=\sqrt{(p-T+1)(p+T+2)T(T+1)/2}\,(p-2\omega+T-1)/3(2\omega+1)(\omega+2)(\omega+1)(\omega-1)(2T+1)$
		T	T	0	$\sqrt{2}\left\{5(p-\omega)^2\omega-5\omega T(T+1)+(\omega+1)(\omega^2+2\omega+2)\right\}/D$
				2	$\left\{(p-\omega)^2(2\omega-3)-(2\omega-3)T(T+1)-(\omega+1)(2\omega^2+\omega-2)\right\}/D$
					$D=3\sqrt{2}\,(2\omega+1)(\omega+1)(\omega-1)$
1	2	T	$T \pm 2$	2	$\sqrt{T(T+2)(p+T+3)(p-T)/3(2T+1)(2T+3)}\,(p-2\omega+T-2)/(\omega+3)(\omega+1)$
		$T \pm 1$	$T \pm 1$		$-\sqrt{T(T+2)/6}\left[-(p-\omega)(p-\omega-2)+T^2+(\omega^2+5\omega+5)\right]/(\omega+1)(\omega+3)(2T+1)$
		$T \mp 1$	$T \pm 1$		$\sqrt{(T+1)(T+2)(p+T+2)(p-T+1)/6}\,(p-2\omega+T-3)/(\omega+1)(\omega+3)(2T+1)$
		T	T		$\left\{(p-\omega)(p-\omega-2)+T(T+1)-(\omega^2+5\omega+6)\right\}/(\omega+1)(\omega+3)\sqrt{2(2T-1)(2T+3)}$

Table C2. As table C1 for the component $V^{(22)}$.

t'	t	T_p'	T_p	u	$((\omega' t')\, p'\, T_p'\, T \Vert \Gamma(V^{(22)}) \Vert (\omega\, t)\, p\, T_p\, T)$
$(\omega' = \omega + 2)$					
0	2	T	$T \pm 2$	2	$\sqrt{2(p-T)(p-T+2)(T+2)(T+1)/5(2T+3)(2T+1)}$
		T	T		$-2\sqrt{(p+T+3)(p-T+2)T(T+1)/15(2T-1)(2T+3)}$
$(\omega' = \omega + 1)$					
0	1	T	$T \pm 1$	1	$\sqrt{2(T+1)(p-T+1)/(2T+1)}\,[2(p-\omega)-T]/3(\omega-1)$
0	2	T	$T \pm 1$	2	$-\sqrt{2T(T+1)(T+2)(p-T+1)/5(2T+1)}\,/(\omega+1)$
1	0	$T \pm 1$	T	1	$-\sqrt{2(T+1)(p+T+3)/3(2T+1)}\,[2(p-\omega)+T]/\omega$
1	1	$T \pm 1$	T	1	$\sqrt{3T(p+T+3)/(2T+1)}\,[2(p-\omega)(2\omega+1)+(\omega-1)+(\omega-1)(2T+1)]/6(\omega+1)(\omega-1)$
				2	$-\sqrt{3T(p+T+3)/5(2T+1)}\,[2(p-\omega)-\omega+1-(\omega-1)(2T+1)]/2(\omega+1)(\omega-1)$
		T	$T \pm 1$	1	$\sqrt{3T(p-T+1)/(2T+1)}\,[2(p-\omega)(2\omega+1)-\omega-2-(\omega+2)(2T+1)]/6(\omega+1)(\omega-1)$
				2	$-\sqrt{3T(p-T+1)/5(2T+1)}\,[2(p-\omega)-\omega-\omega(2T+1)]/2(\omega+1)(\omega-1)$
1	2	$T \pm 1$	$T \pm 2$	1	$F\left\{3(p-\omega)-2(2\omega^2+10\omega+11)T(T+1)-(\omega^2+2\omega-2)+(2T+1)\left[(p-\omega)(4\omega^2+14\omega+7)+(\omega^2+2\omega-2)\right]\right\}$
				2	$-F\left\{(p-\omega)(2\omega-3)-2(2\omega^2+4\omega-3)T(T+1)+(\omega^2-6)+(2T+1)\left[(p-\omega)(4\omega+9)-(\omega^2-6)\right]\right\}$
					$F=\sqrt{2(p-T)(T+2)/5(2T+3)}\,/2(\omega+3)(\omega+1)(\omega-2)(2T+1)$
		$T \mp 1$	$T \pm 2$	1	$-3F$
				2	$(2\omega-3)F$
					$F=\sqrt{2(p+T+3)(p-T+2)(p-T)T(T+1)(T+2)/5(2T+3)}\,/(\omega+3)(\omega+1)(\omega-2)(2T+1)$
		$T \pm 1$	T	1	$F\left\{2(p-\omega)(4\omega^2+14\omega+1)-4(\omega^2-\omega-11)T(T+1)+9(\omega^2+\omega-4)-(2T+1)(\omega+1)[2(p-\omega)(2\omega+5)+3(\omega+4)]\right\}$
				2	$3F\left\{-10(p-\omega)-4(\omega^2+\omega-5)T(T+1)+(5\omega^2+\omega-24)+(2T+1)(\omega+1)[2(p-\omega)+\omega]\right\}$
					$F=\sqrt{(p+T+3)(2T-1)T/15(2T+1)(2T+3)}\,/2(\omega+3)(\omega+1)(\omega-2)(2T-1)$
		T	$T \pm 1$	1	$F\left[2(p-\omega)(2\omega^2+7\omega+2)+(3\omega^2+6\omega-6)-(2T+1)(\omega^2+2\omega-2)\right]$
				2	$-F\left[2(p-\omega)(\omega+6)-3(\omega^2-6)+(2T+1)(\omega^2-6)\right]$
					$F=\sqrt{(p-T+1)(T+2)/5(2T+1)}\,/2(\omega+3)(\omega+1)(\omega-2)$
$(\omega' = \omega)$					
0	0	T	T	0	$\left[8(p-\omega)^2+4T(T+1)-2\omega(\omega+3)\right]/6(\omega-1)\omega$
0	1	T	T	1	$\sqrt{2T(T+1)}\,(p-\omega)/(\omega+1)(\omega-1)$
0	2	T	$T \pm 2$	2	$-\sqrt{(T+1)(T+2)(p+T+3)(p-T)/15(2T+1)(2T+3)}\,\{2(p-\omega)(\omega+1)-(2\omega+5)(\omega+2)+(\omega+4)(2T+1)\}/(2\omega+1)(\omega+3)(\omega+1)$
		T	T		$-\sqrt{2T(T+1)/5(2T-1)(2T+3)}\,\left\{2(\omega^3+7\omega^2+21\omega+24)-2(5\omega+14)T(T+1)-(p-\omega)[2(\omega+1)(p-\omega)-3(2\omega+5)]\right\}/3(2\omega+1)(\omega+3)(\omega+1)$

Table C2. (continued)

t'	t	T'_p	T_p	u	$((\omega'\,t')\,p'\,T'_p\,T\|\Gamma(V^{(22)})\|(\omega\,t)\,p\,T_p\,T)$
$(\omega'=\omega)$					
1	1	T	T	0	$5\left\{4(p-\omega)^2\omega(2\omega-1)+2(2\omega-3)(\omega+1)T(T+1)-(2\omega^3+3\omega^2-7\omega-2)(\omega+1)\right\}/D$
				1	$15\sqrt{3}\left[-2(p-\omega)^2\omega+\omega(\omega+1)-T(T+1)\right]/D$
				2	$\sqrt{5}\left[-2(\omega^2-2\omega-6)(p-\omega)^2-(10\omega^2-20\omega-3)T(T+1)+(2\omega^3-7\omega-8)(\omega+1)\right]/D$
					$D=15(2\omega+1)(\omega+1)(\omega-1)(\omega-2)$
		$T\pm1$	$T\pm1$	0	$10\{2(p-\omega)[(p-\omega)(5\omega+2)-12(2\omega+3)\omega T(T+1)]-2(\omega-2)T(T+1)+2(\omega+2)(\omega+1)(\omega-2)\}/D$
					$+10(2T+1)\left[2(p-\omega)^2(4\omega^3+6\omega^2+\omega+2)+2(2\omega^3+3\omega^2+2\omega+4)T(T+1)-(2\omega^3+3\omega^2-9\omega+2)(\omega+2)(\omega+1)\right]/D$
				1	$-15\sqrt{3}\left\{2(p-\omega)\left[(p-\omega)\omega^2-(2\omega^2-\omega+2)(2\omega+3)T(T+1)\right]+(\omega-2)T(T+1)-(\omega+2)(\omega+1)(\omega-2)\right\}/D$
					$-15\sqrt{3}(2T+1)\left[2(p-\omega)^2(3\omega+4)\omega+(4\omega^2+5\omega-2)T(T+1)-(3\omega-2)(\omega+2)(\omega+1)\right]/D$
				2	$\sqrt{5}\left\{2(p-\omega)\left[(p-\omega)(3\omega^3-4\omega-4)-3(7\omega-6)(2\omega+3)T(T+1)\right]+(6\omega^2+12\omega-5)(\omega-2)T(T+1)-(6\omega^2+12\omega-5)(\omega+2)(\omega+1)(\omega-2)\right\}/D$
					$+\sqrt{5}(2T+1)\left[2(p-\omega)^2(\omega^3+16\omega+20)+(10\omega^3-17\omega+26)T(T+1)-(2\omega^3-15\omega+26)(\omega+2)(\omega+1)\right]/D$
					$D=30(2\omega+1)(\omega+2)(\omega+1)(\omega-1)(\omega-2)(2T+1)$
		$T\mp1$	$T\pm1$	0	$-10F[4(p-\omega)(5\omega+2)+2(2\omega+3)(\omega-2)-2(2T+1)(\omega-2)]$
				1	$15\sqrt{3}\,F\left[4(p-\omega)\omega^2-(2\omega+3)(\omega-2)+(\omega-2)(2T+1)\right]$
				2	$-\sqrt{5}\,F\left[4(p-\omega)(3\omega^3-4\omega-4)-(6\omega^2+12\omega-5)(2\omega+3)(\omega-2)+(2T+1)(6\omega^2+12\omega-5)(\omega-2)\right]$
					$F=\sqrt{(p+T+2)(p-T+1)T(T+1)}\,/30(2\omega+1)(\omega+2)(\omega+1)(\omega-1)(\omega-2)(2T+1)$
1	2	T	$T\pm2$	1	$3\sqrt{3}\,F\{4(p-\omega)\omega-2\omega-5+(2T+1)\}$
				2	$-F\left\{4(p-\omega)(\omega^2+2)-(2\omega^2-5)(2\omega+5)+(2T+1)(2\omega^2-5)\right\}$
					$F=\sqrt{2T(T+2)(p+T+3)(p-T)/15(2T+1)(2T+3)}\,/(2\omega+1)(\omega+3)(\omega+1)(\omega-2)$
		$T\pm1$	$T\pm1$	1	$3\sqrt{3}F\left\{-(2\omega^2+12\omega+13)+2(4\omega+7)T(T+1)+(p-\omega)\left[4\omega(p-\omega)+4\omega^2+4\omega-11\right]-(2T+1)\left[(2\omega+1)+(p-\omega)(4\omega^2+8\omega-1)\right]\right\}$
				2	$-F\left\{-(4\omega^4+20\omega^3+18\omega^2-50\omega-61)+2T(T+1)(10\omega^2+12\omega-19)+(p-\omega)\left[-(8\omega^2-6\omega-59)+4(\omega^2+2)(p-\omega)\right]-(2T+1)[3(p-\omega)(10\omega+11)+(2\omega+1)(2\omega-1)]\right\}$
					$F=\sqrt{T(T+2)/15}\,/2(2\omega+1)(\omega+3)(\omega+1)(\omega-2)(2T+1)$
		$T\mp1$	$T\pm1$	1	$3\sqrt{3}F\{4(p-\omega)\omega-2\omega-11+5(2T+1)\}$
				2	$-F\left\{-(4\omega^3+14\omega^2+2\omega-47)+4(p-\omega)(\omega^2+2)+(2T+1)(2\omega^2+12\omega-17)\right\}$
					$F=\sqrt{(T+1)(T+2)(p-T+1)(p+T+2)/15}/2(2\omega+1)(\omega+3)(\omega+1)(\omega-2)(2T+1)$
		T	T	1	$-\sqrt{3}F\left\{3(\omega+2)(\omega+3)-3T(T+1)+2(p-\omega)\left[-3(2\omega^2+3\omega-3)+(2\omega-1)(2\omega+5)T(T+1)-3(p-\omega)\omega\right]\right\}$
				2	$F\left\{(\omega+2)(\omega+3)(2\omega^2-5)-T(T+1)(2\omega^2-5)-2(p-\omega)\left[-(2\omega^2-9\omega-23)-3T(T+1)(2\omega+5)+(p-\omega)(\omega^2+2)\right]\right\}$
					$F=\sqrt{1/5(2T-1)(2T+3)}/(2\omega+1)(\omega+3)(\omega+1)(\omega-2)$

Table C2. (continued)

t'	t	T'_p	T_p	u	$((\omega'\,t')\,p'\,T'_p\,T \Vert \Gamma(V^{(22)}) \Vert (\omega\,t)\,p\,T_p\,T)$
$(\omega'=\omega)$					
2	2	$T\pm1$	$T\pm1$	0	$2\sqrt{7}\,\{18(\omega-3)(\omega+1)(\omega+2)-18(\omega-3)T(T+1)+6(p-\omega)[8(\omega-1)(2\omega+3)-4(\omega-1)(2\omega+3)T(T+1)+3(p-\omega)(5\omega-3)]\}/D$
					$2\sqrt{7}\,(2T+1)\big[2(p-\omega)^2(4\omega^3+2\omega^2-9\omega+9)+2(2\omega^2+\omega-9)T(T+1)-(2\omega^5+7\omega^4-9\omega^3-45\omega^2+3\omega+18)\big]/D$
				1	$3\sqrt{7}\big\{9(\omega-3)(\omega+1)(\omega+2)-9(\omega-3)T(T+1)-2(p-\omega)\big[2(2\omega^2-3\omega+7)(2\omega+3)-(2\omega^2-3\omega+7)(2\omega+3)T(T+1)+9(\omega^2-\omega+2)(p-\omega)\big]\big\}/D$
					$-3\sqrt{7}(2T+1)\big[2(p-\omega)^2(7\omega^2+5\omega-6)+(4\omega^2+5\omega-3)T(T+1)-(7\omega^3+20\omega^2-21\omega-18)\big]/D$
				2	$\big\{-9(2\omega^2+4\omega+1)(\omega+2)(\omega+1)(\omega-3)+9(2\omega^2+4\omega+1)(\omega-3)T(T+1)+6(p-\omega)\big[2(5\omega+1)(2\omega+3)-(5\omega+1)(2\omega+3)T(T+1)+3(\omega^2-\omega+2)\omega(p-\omega)\big]\big\}/D$
					$(2T+1)\big[(2\omega^5+4\omega^4+3\omega^3-84\omega^2-291\omega-126)-(10\omega^3-10\omega^2-93\omega-45)T(T+1)-2(\omega^3-\omega^2-66\omega-36)(p-\omega)^2\big]/D$
					$D=6\sqrt{7}\,(2\omega+1)(\omega+3)(\omega+1)(\omega-2)(\omega-3)(2T+1)$
		$T\mp1$	$T\pm1$	0	$4\sqrt{7}\,F[(2\omega+3)(\omega-3)+2(p-\omega)(5\omega-3)-(\omega-3)(2T+1)]$
				1	$3\sqrt{7}\,F\big[(2\omega+3)(\omega-3)-4(p-\omega)(\omega^2-\omega+2)-(2T+1)(\omega-3)\big]$
				2	$F\big[-(2\omega^2+4\omega+1)(2\omega+3)(\omega-3)+4(\omega^2-\omega+2)(p-\omega)\omega+(2T+1)(\omega-3)(2\omega^2+4\omega+1)\big]$
					$F=-\sqrt{(T-1)(T+2)(p-T+1)(p+T+2)/7}\,/2(2\omega+1)(\omega+3)(\omega+1)(\omega-2)(\omega-3)(2T+1)$
		$T\pm2$	$T\pm2$	0	$8\sqrt{7}\,(p-\omega)\big\{-6(\omega-1)(\omega+2)(2\omega+3)-6(2\omega+1)^2(2\omega+3)T(T+1)+(p-\omega)\big[(2\omega^4+5\omega^3+5\omega^2+6\omega+18)+2T(T+1)(2\omega+3)(2\omega^3+2\omega^2+2\omega+3)\big]\big\}/D$
					$-\sqrt{7}\,\big\{2(2\omega^6+11\omega^5-\omega^4-75\omega^3-69\omega^2+24\omega+36)+T(T+1)\big[4(4\omega^6+22\omega^5-4\omega^4-155\omega^3-44\omega^2+339\omega+180)-16T(T+1)(2\omega^4+5\omega^3+14\omega^2+33\omega+27)\big]\big\}/D$
					$+\sqrt{7}\,(2T+1)\big\{-4(2\omega^6+11\omega^5-7\omega^4-87\omega^3-15\omega^2+204\omega+144)+8T(T+1)(2\omega^4+5\omega^3+11\omega^2+36\omega+45)\big\}/D$
					$+8\sqrt{7}\,(2T+1)(p-\omega)\big[6(\omega-1)(\omega+2)(2\omega+3)-6T(T+1)(2\omega+3)(2\omega^2+2\omega-1)+(p-\omega)\omega(4\omega^3+10\omega^2+25\omega+33)\big]/D$
				1	$-6\sqrt{7}\,\big\{-(5\omega^4+14\omega^3-16\omega^2-39\omega-18)+T(T+1)\big[-(20\omega^4+84\omega^3-9\omega^2-221\omega-108)+4T(T+1)(8\omega^3+17\omega^2+2\omega-9)\big]\big\}/D$
					$-6\sqrt{7}\,(p-\omega)\big[-(\omega+2)(2\omega+3)(2\omega^2-3\omega+7)-4T(T+1)(2\omega+1)(2\omega+3)(\omega^2+5)+2(p-\omega)(5\omega^3+11\omega^2+8\omega-6)+4(p-\omega)T(T+1)(10\omega^3+22\omega^2+7\omega-3)\big]/D$
					$-6\sqrt{7}\,(2T+1)\big\{-(13\omega^4+40\omega^3-38\omega^2-159\omega-90)+T(T+1)(16\omega^3+37\omega^2+\omega-36)\big\}/D$
					$-6\sqrt{7}\,(2T+1)(p-\omega)\big[(\omega+2)(2\omega+3)(2\omega^2-3\omega+7)-2T(T+1)(\omega+1)(2\omega+3)(2\omega^2-\omega+8)+2(p-\omega)(13\omega^3+25\omega^2+10\omega+6)\big]/D$
				2	$2\big\{-(2\omega^6+8\omega^5-37\omega^4-84\omega^3+234\omega^2+381\omega+126)+T(T+1)\big[-(8\omega^6+32\omega^5-50\omega^4-238\omega^3+317\omega^2+921\omega+360)+4T(T+1)(10\omega^4+10\omega^3-2\omega^2+9\omega+9)\big]\big\}/D$
					$+2(p-\omega)\big[-3(\omega+2)(2\omega+3)(5\omega+1)-84T(T+1)(\omega-1)(2\omega+1)(2\omega+3)+2(p-\omega)(\omega^4+\omega^3+52\omega^2+54\omega+18)+4(p-\omega)T(T+1)(2\omega^4+2\omega^3+77\omega^2+162\omega+63)\big]/D$
					$+2(2T+1)\big\{-(10\omega^6+40\omega^5-65\omega^4-342\omega^3-36\omega^2+429\omega+198)+T(T+1)\omega(26\omega^3+26\omega^2-49\omega-57)\big\}/D$
					$+2(2T+1)\big\{(p-\omega)\big[3(\omega+2)(2\omega+3)(5\omega+1)-6T(T+1)(2\omega+3)(11\omega^2-\omega-4)+2(p-\omega)(5\omega^4+5\omega^3+86\omega^2+156\omega+54)\big]\big\}/D$
					$D=6\sqrt{7}\,(2\omega+1)(\omega+3)(\omega+2)(\omega+1)(\omega-2)(\omega-3)(2T+1)(2T+3)$

Table C2. (continued)

t'	t	T'_p	T_p	u	$((\omega' t')\, p'\, T'_p\, T \Vert \Gamma(V^{(22)}) \Vert (\omega\, t)\, p\, T_p\, T)$
$(\omega'=\omega)$					
2	2	T	$T\pm 2$	0	$4\sqrt{7}\,F\{2(p-\omega)(10\omega^2+14\omega-15)+4(\omega^2+5\omega+3)T(T+1)+(4\omega^3-\omega^2-39\omega-36)-(2T+1)[2(p-\omega)(5\omega^2+7\omega-3)+(\omega-1)\omega(2\omega+3)]\}$
				1	$3\sqrt{7}\,F\{-4(p-\omega)(2\omega^3+2\omega^2-\omega+9)-12\omega(\omega+1)T(T+1)+(4\omega^3+7\omega^2-23\omega-30)+(2T+1)[4(p-\omega)(\omega^3+\omega^2+\omega+3)-(2\omega+3)(\omega^2-5\omega-2)]\}$
				2	$F\{4(p-\omega)(2\omega^4+2\omega^3-3\omega^2+14\omega+3)-4(2\omega^4-10\omega^3-9\omega^2+23\omega+12)T(T+1)-(8\omega^5+14\omega^4-30\omega^3-157\omega^2-195\omega-54)\}$
					$+F(2T+1)[-4(p-\omega)(\omega^3+2\omega^2+5\omega+3)(\omega-1)+(2\omega^4+2\omega^3-27\omega^2-\omega+6)(2\omega+3)]$
					$F=\sqrt{T(T+2)(p+T+3)(p-T)(2T-1)/42(2T+1)}\,/(\omega-3)(\omega-2)(\omega+1)(\omega+2)(\omega+3)(2\omega+1)(2T-1)(2T+3)$
		$T\mp 2$	$T\pm 2$	0	$2F$
				1	$-(\omega-1)F$
				2	$(\omega^2-2\omega-1)F/\sqrt{7}$
					$F=12\sqrt{(T-1)T(T+1)(T+2)(p-T)(p-T+2)(p+T+1)(p+T+3)/(2T-1)(2T+3)}\,/(2\omega+1)(\omega+3)(\omega+2)(\omega+1)(\omega-2)(\omega-3)(2T+1)$
		$T\pm 2$	T	0	$4\sqrt{7}\,F\{2(p-\omega)(10\omega^2+14\omega-15)-4(\omega^2-7\omega-15)T(T+1)+(4\omega^2-7\omega-24)(\omega+3)-(2T+1)[2(5\omega^2+7\omega-3)(p-\omega)+(2\omega+3)\omega(\omega-1)]\}$
				1	$3\sqrt{7}\,F\{-4(2\omega^3+2\omega^2-\omega+9)(p-\omega)-4(5\omega^2+\omega-12)T(T+1)+(4\omega^3+13\omega^2-29\omega-66)+(2T+1)[4(p-\omega)(\omega^3+\omega^2+\omega+3)-(\omega^2-5\omega-2)(2\omega+3)]\}$
				2	$F\{4(p-\omega)(2\omega^4+2\omega^3-3\omega^2+14\omega+3)+4(2\omega^4+14\omega^3-21\omega^2-73\omega-24)T(T+1)-(8\omega^5+26\omega^4-18\omega^3-247\omega^2-345\omega-90)\}$
					$-F(2T+1)[4(\omega^3+2\omega^2+5\omega+3)(\omega-1)(p-\omega)-(2\omega^4+2\omega^3-27\omega^2-\omega+6)(2\omega+3)]$
					$F=\sqrt{T(T+2)(p+T+3)(p-T)(2T-1)/42(2T+1)}\,/(\omega-3)(\omega-2)(\omega+1)(\omega+2)(\omega+3)(2\omega+1)(2T-1)(2T+3)$
		T	T	0	$4\sqrt{7}(p-\omega)\{(p-\omega)[4(2\omega^4+5\omega^3-5\omega^2-8\omega+15)T(T+1)-3(2\omega^2+7\omega-6)(\omega+2)(\omega-3)]+36(2\omega+3)T(T+1)+36(\omega+2)(\omega-1)(2\omega+3)\}/D$
					$+\sqrt{7}\{2T(T+1)[4(2\omega^4+5\omega^3-14\omega^2-35\omega-12)T(T+1)-(4\omega^6+22\omega^5+20\omega^4-103\omega^3-372\omega^2-489\omega-234)]+3(\omega+2)(2\omega^5+7\omega^4+9\omega^3-45\omega^2-123\omega-90)\}/D$
				1	$-18\sqrt{7}(p-\omega)[4(p-\omega)(\omega^3+3\omega^2+\omega-3)T(T+1)+(p-\omega)(\omega+2)(\omega-3)(\omega-4)+4(2\omega+3)(\omega-1)T(T+1)+(2\omega^2-3\omega+7)(2\omega+3)(\omega+2)]/D$
					$-9\sqrt{7}\{T(T+1)[4(\omega+1)T(T+1)\omega-(4\omega^4+8\omega^3-5\omega^2-15\omega+6)]-(\omega^3-10\omega^2-21\omega-18)(\omega+2)\}/D$
				2	$-2(p-\omega)[4(p-\omega)(\omega^4+\omega^3-37\omega^2-76\omega-33)T(T+1)-3(p-\omega)(\omega+2)(\omega-3)(5\omega^2+10\omega+6)-36(\omega^2-2\omega-1)(2\omega+3)T(T+1)-9(5\omega+1)(2\omega+3)(\omega+2)]/D$
					$-\{T(T+1)[4(10\omega^4+10\omega^3-73\omega^2-157\omega-60)T(T+1)-(8\omega^6+32\omega^5+154\omega^4-146\omega^3-1869\omega^2-2823\omega-954)]+3(10\omega^5+20\omega^4-57\omega^3-222\omega^2-285\omega-90)(\omega+2)\}/D$
					$D=3\sqrt{7}(2\omega+1)(\omega+3)(\omega+2)(\omega+1)(\omega-2)(\omega-3)(2T-1)(2T+3)$

dimensional, are given below.

$$KK^\dagger(p(\omega 2)T)_{T_p'T_p} = \frac{(\omega-2)!(2\omega+1)!!M_{T_p'T_p}}{(\omega-k+1-T)!(2T+1)(2\omega+3-2k)!!2^{k-1}}$$

where $p = T+2k$, $k = 0, 1, 2, \ldots$, and

$$\begin{aligned} M_{T-2,T-2} = \frac{1}{4(2T-1)} \{ & -(2k\omega+12k-2\omega^2-5\omega+6)(2k-2\omega-3)(\omega+1) \\ & -(6k+2\omega+11)(2k-2\omega-1)(2\omega+3)T+(8k^2\omega+20k^2 \\ & -16k\omega^2-48k\omega-32k+8\omega^3+28\omega^2+18\omega-17)(2\omega+3)T^2 \\ & +4(2k-2\omega-1)(2\omega+3)(2\omega+5)T^3+4(2\omega+3)(2\omega+5)T^4\} \end{aligned}$$

$$\begin{aligned} M_{T-2,T} = \frac{1}{2(2T-1)} & \sqrt{\frac{3(T+1)(k+1)(2T+2k+1)(2T+1)(T-1)}{(2T+3)}} \\ & \times \{3(2\omega+3-2k)(\omega+1)+(2\omega-2k-1)(2\omega+3) \\ & \times T-2(2\omega+3)T^2\} \end{aligned}$$

$$M_{T-2,T+2} = \frac{3}{2}\sqrt{\frac{k(k+1)(T-1)T(T+1)(T+2)(2T+2k+1)(2T+2k+3)}{(2T-1)(2T+3)}}$$

$$\begin{aligned} M_{T,T} = \frac{1}{2(2T-1)(2T+3)} \{ & -3(k\omega-3k-\omega^2-2\omega-3)(2k-2\omega-3)(\omega+1) \\ & -(4k^2\omega^2-40k^2\omega-48k^2-8k\omega^3+28k\omega^2+40k\omega-6k+4\omega^4+12\omega^3 \\ & +53\omega^2+96\omega+45)T+2(12k^2\omega^2+24k^2\omega+18k^2-24k\omega^3-80k\omega^2 \\ & -40k\omega+33k+12\omega^4+56\omega^3+73\omega^2+36\omega+18)T^2+4(4k^2\omega^2 \\ & +8k^2\omega+6k^2-8k\omega^3-20k\omega^2-8k\omega+12k+4\omega^4 \\ & +12\omega^3+5\omega^2+9)T^3+8((2\omega^2+4\omega+3)k-2\omega^3 \\ & -7\omega^2-6\omega)T^4\} \end{aligned}$$

$$\begin{aligned} M_{T,T+2} = \frac{1}{(2T+3)} & \sqrt{\frac{3kT(T+2)(2T+1)(2T+2k+3)}{(2T-1)}} \\ & \times \{(k\omega-\omega^2-3\omega-3)-(2k\omega+3k-2\omega^2-6\omega-3)T\} \end{aligned}$$

$$\begin{aligned} M_{T+2,T+2} = \frac{1}{(2T+3)} \{ & 3(k-\omega)(k-\omega-1)(\omega+2)(\omega+3) \\ & +(8k^2\omega^2+38k^2\omega+39k^2-16k\omega^3-86k\omega^2-130k\omega-51k \\ & +8\omega^4+48\omega^3+88\omega^2+48\omega)T+(4k^2\omega^2+16k^2\omega+15k^2 \\ & -8k\omega^3-40k\omega^2-56k\omega-21k+4\omega^4+24\omega^3+44\omega^2+24\omega)T^2\}. \end{aligned}$$

The expressions show the same symmetry under the transformation $T \to -(T+1)$ that was discussed in appendix C. However, since we have followed [11, 12] by using

$k = \frac{1}{2}(p-T)$ instead of p, the transformation must also include $k \rightarrow k+T+\frac{1}{2}$. In this way, $KK^{\dagger}_{T+2,T+2} \rightarrow KK^{\dagger}_{T-2,T-2}$ and $KK^{\dagger}_{T+2,T} \rightarrow KK^{\dagger}_{T-2,T}$ while $KK^{\dagger}_{T-2,T+2}$ is invariant because the matrix is symmetric.

References

[1] Hecht K T 1965 *Phys. Rev.* B **139** 794
[2] Flowers B H and Szpikowski S 1965 *Proc. Phys. Soc.* **86** 672
[3] Hecht K T 1967 *Nucl. Phys.* A **102** 11
[4] Hemenger R P and Hecht K T 1970 *Nucl. Phys.* A **145** 468
[5] Evans J A, Elliott J P and Szpikowski S 1985 *Nucl. Phys.* A **435** 317
[6] Rowe D J 1984 *J. Math. Phys.* **25** 2662
[7] Deenan J and Quesne C 1984 *J. Math. Phys.* **25** 1638, 2354
[8] Rowe D J, Rosensteel G and Gilmore R 1985 *J. Math. Phys.* **26** 2787
[9] Rowe D J, Le Blanc R and Hecht K T 19888 *J. Math. Phys.* **29** 287
[10] Hecht K T 1987 *The Vector Coherent State Method and its Application to Problems of Higher Symmetries (Lecture Notes in Physics 290)* (Berlin: Springer)
[11] Hecht K T and Elliott J P 1985 *Nucl. Phys.* A **438** 29
[12] Hecht K T 1989 *Nucl. Phys.* A **493** 29
[13] Flowers B H 1952 *Proc. R. Soc.* A **212** 248
[14] Thompson M J, Elliott J P and Evans J A 1987 *Phys. Lett.* **195B** 511
[15] Elliott J P 1985 *Rep. Prog. Phys.* **48** 171
[16] Elliott J P 1990 *Prog. Part. Nucl. Phys.* **25** 325
[17] Hamilton W D, Irbäck A and Elliott J P 1984 *Phys. Rev. Lett.* **53** 2469
[18] Richter A 1990 *Nucl. Phys.* A **597** 99c
[19] Biedenharn L C and Louck J D 1981 *Encyclopedia of Mathematics and Applications* vol 9 (Reading, MA: Addison-Wesley)
[20] Raynal J 1978 *J. Math. Phys.* **19** 467